Math 111 Final Exam Review KEY

1. Use the graph of $y = f(x)$ in Figure 1 to answer the following. Approximate where necessary.

(a) Evaluate $f(-1)$.
$f(-1) = 0$

(b) Evaluate $f(0)$.
$f(0) = -6$

(c) Solve $f(x) = 0$.
$x = -2, x = -1, x = 2, \text{or } x = 3$
Solution Set: $\{-2, -1, 2, 3\}$

(d) Solve $f(x) = -7$.
$x \approx -2.5, x \approx 0.5, \text{ or } x \approx 3.5$
Solution Set: $\{x \,|\, x \approx -2.5, 0.5, 3.5\}$

(e) Determine if f is even, odd, or neither from its graph.
Neither. The function is not symmetric about the y-axis and is therefore not even. The function is not symmetric about the origin and is therefore not odd.

(f) State any local maximums or local minimums.
There is a local maximum of 2 at about -1.5 and a local maximum of 2 at about 2.5. There is a local minimum of -7 at about 0.5.

(g) State the domain and range of f.
Domain: $(-\infty, \infty)$
Range: $(-\infty, 2]$

(h) Over what interval(s) is the function increasing?
$(-\infty, -1.5) \cup (0.5, 2.5)$

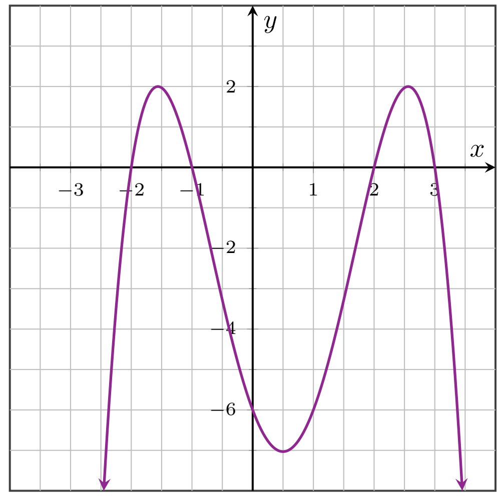
(i) Over what interval(s) is the function decreasing?
$(-1.5, 0.5) \cup (2.5, \infty)$

(j) Over what interval(s) is the function concave up?
Concave up: $(-0.5, 1.5)$

(k) Over what interval(s) is the function concave down?
Concave down: $(-\infty, -0.5) \cup (1.5, \infty)$

(l) Find the zeros of f.
The zeros are -2, -1, 2, and 3.

FIGURE 1



(m) Find a possible formula for this polynomial function.
To find a formula, we note that since the zeros of the function are -2, -1, 2, and 3, the function will have $(x+2), (x+1), (x-2)$ and $(x-3)$ as factors. Since the graph of the function goes straight through each of its zeros, none of these factors repeat. Therefore a possible function is $f(x) = k(x+2)(x+1)(x-2)(x-3)$. As the graph contains the point $(0, -6)$, we know that $f(0) = -6$. We will use this to find k:

$$-6 = k(0+2)(0+1)(0-2)(0-3)$$
$$-\frac{1}{2} = k$$

Therefore a possible formula for this polynomial function is $f(x) = -\frac{1}{2}(x+2)(x+1)(x-2)(x-3)$.

2. Let $f(x) = \dfrac{2x-6}{x+4}$.

(a) Find $f^{-1}(x)$.

$$y = \frac{2x-6}{x+4}$$

Finding the inverse :

$$\begin{aligned}
x &= \frac{2y-6}{y+4} \\
x(y+4) &= 2y-6 \\
xy+4x &= 2y-6 \\
xy+4x-4x &= 2y-6-4x \\
xy &= 2y-6-4x \\
xy-2y &= 2y-6-4x-2y \\
xy-2y &= -6-4x \\
y(x-2) &= -4x-6 \\
y &= \frac{-4x-6}{x-2}
\end{aligned}$$

Therefore $f^{-1}(x) = \dfrac{-4x-6}{x-2}$. This can be simplified to $f^{-1}(x) = -\dfrac{4x+6}{x-2}$.

(b) Confirm the inverse by computing $f^{-1}\left(f(x)\right)$ and $f\left(f^{-1}(x)\right)$.

$$\begin{aligned}
f^{-1}\left(f(x)\right) &= f^{-1}\left(f(x)\right) \\
&= f^{-1}\left(\frac{2x-6}{x+4}\right) \\
&= -\frac{4\left(\frac{2x-6}{x+4}\right)+6}{\left(\frac{2x-6}{x+4}\right)-2} \\
&= -\frac{4\left(\frac{2x-6}{x+4}\right)+6}{\left(\frac{2x-6}{x+4}\right)-2}\cdot\frac{x+4}{x+4} \\
&= -\frac{4(2x-6)+6(x+4)}{(2x-6)-2(x+4)} \\
&= -\frac{8x-24+6x+24}{2x-6-2x-8} \\
&= -\frac{14x}{-14} \\
&= x
\end{aligned}$$

$$\begin{aligned}
f\left(f^{-1}(x)\right) &= f\left(f^{-1}(x)\right) \\
&= f\left(-\frac{4x+6}{x-2}\right) \\
&= \frac{2\left(-\frac{4x+6}{x-2}\right)-6}{\left(-\frac{4x+6}{x-2}\right)+4} \\
&= \frac{2\left(-\frac{4x+6}{x-2}\right)-6}{\left(-\frac{4x+6}{x-2}\right)+4}\cdot\frac{x-2}{x-2} \\
&= \frac{-2(4x+6)-6(x-2)}{-(4x+6)+4(x-2)} \\
&= \frac{-8x-13-6x+12}{-4x-6+4x-8} \\
&= \frac{-14x}{(-14)} \\
&= x
\end{aligned}$$

(c) State the domain and range of f and f^{-1}.

Domain of f: $\{x \mid x \neq -4\}$
Range of f: $\{y \mid y \neq 2\}$

Domain of f^{-1}: $\{x \mid x \neq 2\}$
Range of f^{-1}: $\{y \mid y \neq -4\}$

(d) Evaluate $f(0)$.
$f(0) = -\frac{3}{2}$

(e) Solve $f(x) = 3$.

$$\frac{2x-6}{x+4} = 3$$
$$2x - 6 = 3(x+4)$$
$$2x - 6 = 3x + 12$$
$$-18 = x$$

The solution is -18. Solution Set: $\{-18\}$.

(f) Algebraically determine if f is even, odd, or neither.
To show that f is even, it must be shown that $f(-x) = f(x)$. To show that f is odd, it must be shown that $f(-x) = -f(x)$. As

$$f(-x) = \frac{2(-x)-6}{x+4} = \frac{-2x-6}{x+4}$$

$$-f(x) = -\frac{2x-6}{x+4} = \frac{-2x+6}{x+4}$$

and thus $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, it holds that f is neither even or odd.

(g) State any horizontal and vertical asymptotes of f.
There is a horizontal asymptote of $y = 2$ since the ratio of leading terms is 2. There is a vertical asymptote of $x = -4$ since the factor $(x+4)$ appears in the denominator.

(h) State any horizontal and vertical intercepts of f.
The horizontal intercept occurs where $f(x) = 0$ and is $(3, 0)$. The vertical intercept occurs where $x = 0$ and is $\left(0, -\frac{3}{2}\right)$.

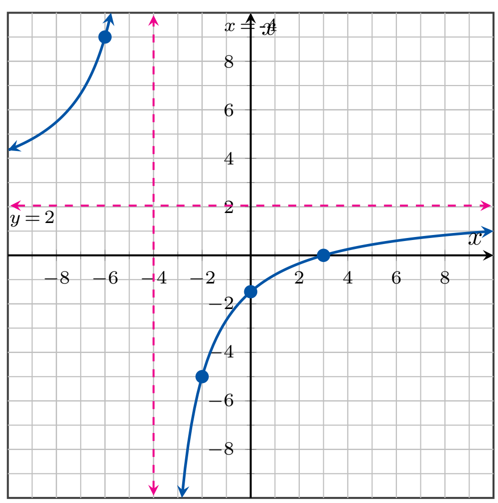
(i) Sketch a graph of $y = f(x)$ in Figure 2.

- The vertical asymptote is $x = -4$.
- The horizontal asymptote is $y = 2$.
- The x-intercept is $(3, 0)$. The only zero is 3.
- The y-intercept is $(0, -1.5)$.

TABLE 1. Behavior of R

Zeros/ VA			
Interval	$(-\infty, -4)$	$(-4, 3)$	$(3, \infty)$
x	-6	-2	10
$R(x)$	$R(-6) = 9$	$R(-2) = -5$	$R(10) = 1$
$+/-$	+ (above)	- (below)	+ (above)
Point	$(-6, 9)$	$(-2, -5)$	$(10, 1)$

FIGURE 2



3. Let $f(x) = |x|$. For each of the following, sketch a graph of the transformation in Figure 4 and write the simplified formula for the function. Describe the order of transformations, being as specific as possible and listing them in an appropriate order.

(a) $y = -f(x)$ Reflect across the x-axis.

(b) $y = 2f(x)$ Stretch vertically by a factor of 2.

(c) $y = f(x) + 3$ Shift up 3 units.

(d) $y = 2f(x+1)+3$ Stretch vertically by a factor of 2, shift left 1 unit, and then shift up 3 units.

(e) $y = f(3x)$ Compress horizontally by a factor of $\frac{1}{3}$.

(f) $y = 3f(-2(x+4)) - 1$ Stretch vertically by a factor of 3, reflect across the y-axis, compress horizontally by a factor of 1/2, shift left 4 units, and shift down 1 unit.

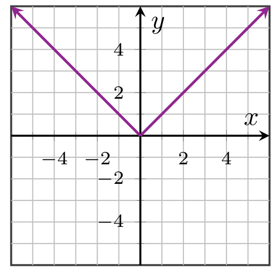
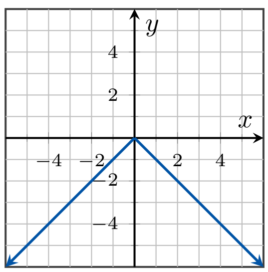
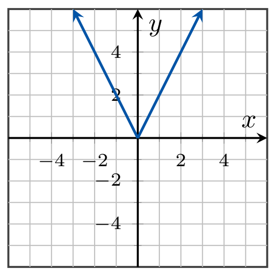


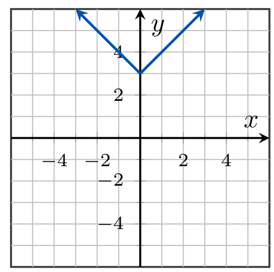
FIGURE 3. Graph of $y = |x|$



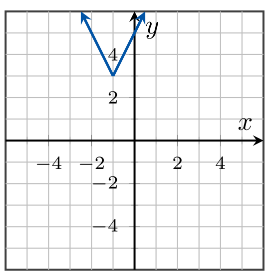
(a) $y = -|x|$



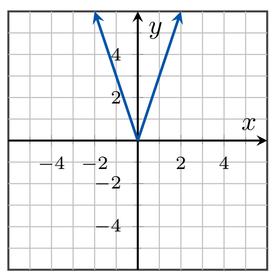
(b) $y = 2|x|$



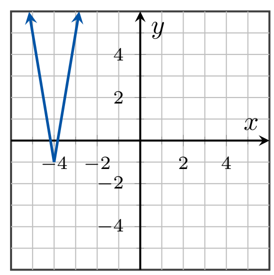
(c) $y = |x| + 3$



(d) $y = 2|x+1| + 3$



(e) $y = |3x|$



(f) $y = 3|-2(x+4)| - 1$

FIGURE 4

4. For each function below, identify the original (or basic) function and explain how the graph is a transformation of the graph of the original function. State all steps to this transformation in an appropriate order.

(a) $g(x) = -\frac{1}{3}\left(4(x-7)\right)^3 - 2$ The parent function is $f(x) = x^3$. Steps to the transformation:

(1) $A = -\frac{1}{3}$: Reflect across the x-axis and compress vertically by a factor of $\frac{1}{3}$.

(2) $B = 4$: Compress horizontally by a factor of $\frac{1}{4}$.

(3) $h = -7$: Shift right 7 units.

(4) $k = -2$: Shift down 2 units.

(Note: Ordering BhAK also appropriate.)

(b) $g(x) = 7\ln(x+4) + 5$ The parent function is $f(x) = \ln(x)$. Steps to the transformation:

(1) $A = 7$: Stretch vertically by a factor of 7.

(2) $h = 4$: Shift left 4 units.

(3) $k = 5$: Shift up 5 units.

(c) $g(x) = \sqrt{-\frac{1}{4}(x+1)} - 3$ The parent function is $f(x) = \sqrt{x}$. Steps to the transformation:

(1) $B = -\frac{1}{4}$: Reflect across the y-axis and stretch horizontally by a factor of 4.

(2) $h = 1$: Shift left 1 unit.

(3) $k = -3$: Shift down 5 units.

5. The point $(-4, 16)$ is on the graph of $y = f(x)$. Determine the point on the graph of...

(a) $y = f(x-5) - 7$

(1) $h = -5$: Shift 5 units right. $(-4, 16) \to (1, 16)$

(2) $k = -7$: Shift 7 units down. $(1, 16) \to (1, 9)$

The point $(1, 9)$ is on the graph of $y = f(x-5) - 7$.

(b) $y = -f(4x)$

(1) $A = -1$: Reflect across the x-axis. $(-4, 16) \to (-4, -16)$

(2) $B = 4$: Compress horizontally by a factor of $\frac{1}{4}$. $(-4, -16) \to (-1, -16)$

The point $(-1, -16)$ is on the graph of $y = -f(4x)$.

(c) $y = 3f(-x)$

(1) $A = 3$: Stretch vertically by a factor of 3. $(-4, 16) \to (-4, 48)$

(2) $B = -1$: Reflect across the y-axis. $(-4, 48) \to (4, 48)$

The point $(4, 48)$ is on the graph of $y = 3f(-x)$.

(d) $y = -\frac{1}{8}f(2(x+3)) + 5$

(1) $A = -\frac{1}{8}$: Reflect across the x-axis and compress vertically by a factor of $\frac{1}{8}$. $(-4, 16) \to (-4, -16) \to (-4, -2)$

(2) $B = 2$: Compress horizontally by a factor of $\frac{1}{2}$. $(-4, -2) \to (-2, -2)$

(3) $h = 3$: Shift left 3 units. $(-2, -2) \to (-5, -2)$.

(4) $k = 5$: Shift up 8 units. $(-5, -2) \to (-5, 3)$.

The point $(-5, 3)$ is on the graph of $y = -\frac{1}{8}f(2(x+3)) + 5$.

6. Complete Table 2 below using the given values in the table. If any value is undefined, write "undefined."

TABLE 2

x	-4	-2	1	2	8
$f(x)$	8	0	-2	-4	-5
$g(x)$	3	4	5	6	3
$(g \cdot f)(x)$	24	0	-10	-24	-15
$(g \circ f)(x)$	-3	und.	4	3	und.
$f(x)+g(x)$	11	4	3	2	-2
$\frac{f(x)}{g(x)}$	$\frac{8}{3}$	0	$-\frac{2}{5}$	$-\frac{2}{3}$	$-\frac{5}{3}$
$f^{-1}(x)$	2	1	und.	und.	-4

7. Find a formula for the piecewise-defined function graphed in Figure 5 below.

$$f(x) = \begin{cases} 2x+5, & x \le -1 \\ 4, & -1 < x < 2 \\ \frac{1}{2}x - 2, & x \ge 2 \end{cases}$$

8. In Figure 6, graph the piecewise function defined by

$$f(x) = \begin{cases} x^2-4, & -3 \le x < 0 \\ 2, & 0 < x < 1 \\ -\frac{1}{2}x + 2, & x \ge 1 \end{cases}$$

FIGURE 5. Graph of $y = f(x)$

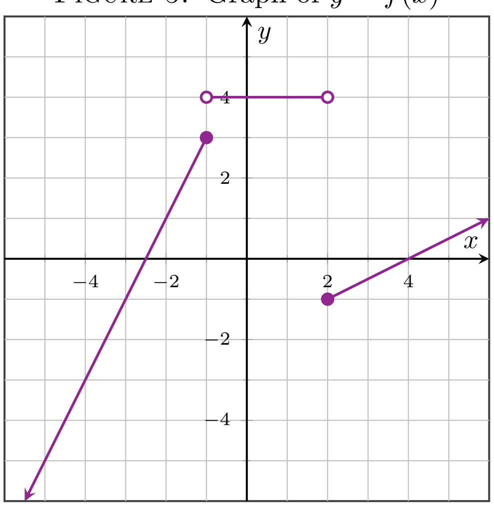
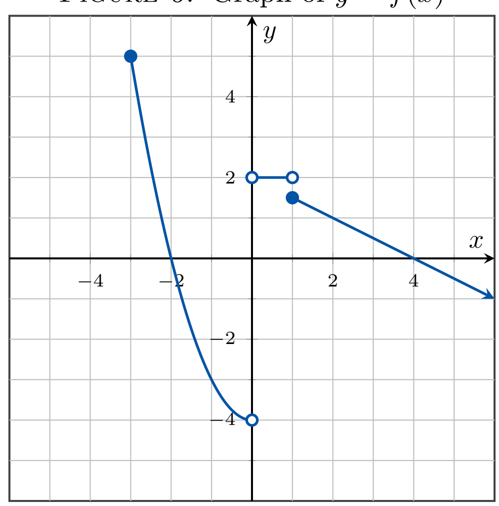


FIGURE 6. Graph of $y = f(x)$



9. The volume, $V(r)$ (in cubic centimeters) of a circular balloon of radius r (in centimeters) is given by $V(r) = \frac{4}{3}\pi r^3$. As someone blows air into the balloon, the radius of the balloon as a function of time t (in seconds) is given by $r = g(t) = 2t$.

(a) Find and interpret $V(3)$.

$$\begin{aligned} V(3) &= \frac{4}{3}\pi(3)^3 \\ &= 36\pi \\ &\approx 113.1 \end{aligned}$$

The volume of a balloon with a radius of 3 centimeters is approximately 113.1 cm^3.

(b) Find and interpret $g(3)$.

$$\begin{aligned} g(3) &= 2(3) \\ &= 6 \end{aligned}$$

A balloon that has air blown into it for 3 seconds will have a radius of 6 cm.

(c) Find and interpret $V(g(3))$.

$$\begin{aligned} V(g(3)) &= V(6) \\ &= \frac{4}{3}\pi(6)^3 \\ &= 288\pi \\ &\approx 904.8 \end{aligned}$$

The volume of a balloon that has been blown into for 3 seconds is about 904.8 cm^3.

(d) Find and interpret $V(g(t))$.

$$\begin{aligned} V(g(t)) &= V(2t) \\ &= \frac{4}{3}\pi(2t)^3 \\ &= \frac{4}{3}\pi\left(8t^3\right) \\ &= \frac{32}{3}\pi t^3 \end{aligned}$$

This function represents the volume of the balloon that has had air blown into it for t seconds.

(e) Explain why $g(V(r))$ in nonsense.
The unit of the input of V is seconds. To input a variable whose unit is centimeters into the function V does not make sense.

10. Find the following for the functions f, g, and h defined by

$$f(x) = \frac{2}{3x+1} \qquad g(x) = 3x^2 + 1 \qquad h(x) = 2x - 5$$

(a) $f(g(2))$

$$\begin{aligned} f\big(g(2)\big) &= f\left(3(2)^2 + 1\right) \\ &= f(13) \\ &= \frac{2}{3(13)+1} \\ &= \frac{2}{40} \\ &= \frac{1}{20} \end{aligned}$$

(b) $(h \circ f)(1)$

$$\begin{aligned}(h \circ f)(1) &= h(f(1)) \\ &= h\left(\frac{2}{3(1)+1}\right) \\ &= h\left(\frac{1}{2}\right) \\ &= 2\left(\frac{1}{2}\right) - 5 \\ &= -4\end{aligned}$$

(c) $(h + g)(1)$

$$\begin{aligned}(h + g)(1) &= h(1) + g(1) \\ &= (2(1) - 5) + \left(3(1)^2 + 1\right) \\ &= (-3) + (4) \\ &= 1\end{aligned}$$

(d) $(g \circ g)(0)$

$$\begin{aligned}(g \circ g)(0) &= g\big(g(0)\big) \\ &= g\big(3(0)^2 + 1\big) \\ &= g(1) \\ &= 3(1)^2 + 1 \\ &= 4\end{aligned}$$

(e) $(f - g)(0)$

$$\begin{aligned}(f - g)(0) &= f(0) - g(0) \\ &= \frac{2}{3(0)+1} - \left(3(0)^2 + 1\right) \\ &= 1\end{aligned}$$

(f) $(g \cdot g)(x)$

$$\begin{aligned}(g \cdot g)(x) &= g(x) \cdot g(x) \\ &= \left(3x^2 + 1\right)\left(3x^2 + 1\right) \\ &= 9x^4 + 6x^2 + 1\end{aligned}$$

(g) $(f \circ g)(x)$

$$\begin{aligned}(f \circ g)(x) &= f\big(g(x)\big) \\ &= f\big(3x^2+1\big) \\ &= \frac{2}{3\big(3x^2+1\big)+1} \\ &= \frac{2}{9x^2+3+1} \\ &= \frac{2}{9x^2+4}\end{aligned}$$

(h) $(g \circ h)(x)$

$$\begin{aligned}(g \circ h)(x) &= g\big(h(x)\big) \\ &= g\big(2x-5\big) \\ &= 3(2x-5)^2+1 \\ &= 3(4x^2-20x+25)+1 \\ &= 12x^2-60x+75+1 \\ &= 12x^2-60x+76\end{aligned}$$

(i) $(h \circ h)(x)$

$$\begin{aligned}(h \circ h)(x) &= h\big(h(x)\big) \\ &= h\big(2x-5\big) \\ &= 2(2x-5)-5 \\ &= 4x-10-5 \\ &= 4x-15\end{aligned}$$

11. Find the algebraic rule (or formula) of an exponential function that passes through each pair of points:

(a) $\left(-1, \frac{1}{3}\right)$ and $(1, 12)$

We will find C and a for $f(x) = Ca^x$. As f contains the points $\left(-1, \frac{1}{3}\right)$ and $(1, 12)$, we have $\dfrac{1}{3} = Ca^{-1}$ and $12 = Ca^1$.

One method is to make a ratio in order to eliminate C:

$$\begin{aligned}\frac{12}{\frac{1}{3}} &= \frac{Ca}{Ca^{-1}} \\ 36 &= a^2 \\ \pm 6 &= a\end{aligned}$$

Since a must be positive, $a = 6$. Substituting to find C, we obtain

$$\begin{aligned}C(6)^1 &= 12 \\ C &= 2\end{aligned}$$

The exponential function passing through these points is defined by $f(x) = 2(6)^x$.

(b) $(2, 128)$ and $(5, 2)$

We will find C and a for $f(x) = Ca^x$. As f contains the points $(2, 128)$ and $(5, 2)$, we have $128 = Ca^2$ and $2 = Ca^5$. Thus

$$\begin{aligned} \frac{2}{128} &= \frac{Ca^5}{Ca^2} \\ \frac{1}{64} &= a^3 \\ \frac{1}{4} &= a \end{aligned}$$

Substituting to find C:

$$\begin{aligned} C\left(\frac{1}{4}\right)^2 &= 128 \\ C &= 128 \cdot 16 \\ &= 2048 \end{aligned}$$

The exponential function passing through these points is defined by $f(x) = 2048\left(\frac{1}{4}\right)^x$.

12. Find the exact values of the expressions below.

(a) $\log_4(64) = 3$

(b) $\ln\left(\sqrt{e}\right) = \frac{1}{2}$

(c) $\log_{10}\left(\frac{1}{100}\right) = -2$

13. Solve the following equations. Give the exact solution and then round accurately to two decimal places. Clearly state each solution set.

(a) $7^x - 1 = 4$

$$\begin{aligned} 7^x - 1 &= 4 \\ 7^x &= 5 \\ x &= \log_7(5) \\ x &\approx 0.83 \end{aligned}$$

The solution set is $\{\log_7(5)\}$. Equivalent solution sets are $\left\{\frac{\ln(5)}{\ln(7)}\right\}$ and $\left\{\frac{\log(5)}{\log(7)}\right\}$.

(b) $e^{5x} = 10$

$$\begin{aligned} e^{5x} &= 10 \\ 5x &= \ln(10) \\ x &= \frac{\ln(10)}{5} \\ x &\approx 0.46 \end{aligned}$$

The solution set is $\left\{\frac{\ln(10)}{5}\right\}$.

(c) $5e^x = 10$

$$\begin{aligned} 5e^x &= 10 \\ e^x &= 2 \\ x &= \ln(2) \\ x &\approx 0.69 \end{aligned}$$

The solution set is $\{\ln(2)\}$.

(d) $3^{x^2} = 9^{x+4}$

$$\begin{aligned} 3^{x^2} &= 9^{x+4} \\ 3^{x^2} &= \left(3^2\right)^{x+4} \\ 3^{x^2} &= 3^{2(x+4)} \\ x^2 &= 2(x+4) \\ x^2 &= 2x+8 \\ x^2-2x-8 &= 0 \\ (x-4)(x+2) &= 0 \\ x-4=0 \quad &\text{or} \quad x+2=0 \\ x=4 \quad &\text{or} \quad x=-2 \end{aligned}$$

Both solutions check. The solution set is $\{-2, 4\}$.

(e) $3^{2x+1} = 6$

One Approach:

$$\begin{aligned} 3^{2x+1} &= 6 \\ \log\left(3^{2x+1}\right) &= \log(6) \\ (2x+1)\log(3) &= \log(6) \\ 2x\log(3)+\log(3) &= \log(6) \\ 2x\log(3) &= \log(6)-\log(3) \\ x &= \frac{\log(6)-\log(3)}{2\log(3)} \\ \text{or} \quad x &= \frac{\log(2)}{\log(9)} \end{aligned}$$

An Alternate Approach:

$$\begin{aligned} 3^{2x+1} &= 6 \\ \log_3(6) &= 2x+1 \\ \log_3(6)-1 &= 2x \\ \frac{\log_3(6)-1}{2} &= \frac{2x}{x} \\ x &= \frac{\log_3(6)-1}{2} \\ x &=\approx 0.315 \end{aligned}$$

The solution set is $\left\{\frac{\log(6)-\log(3)}{2\log(3)}\right\}$ or $\left\{\frac{\log(2)}{\log(9)}\right\}$ or $\{\log_2(9)\}$ or $\left\{\frac{\log_3(6)-1}{2}\right\}$.

(f) $5 \cdot 7^x = 3 \cdot 2^{x-7}$

$$\begin{aligned}
5 \cdot 7^x &= 3 \cdot 2^{x-7} \\
\log(5 \cdot 7^x) &= \log(3 \cdot 2^{x-7}) \\
\log(5) + \log(7^x) &= \log(3) + \log(2^{x-7}) \\
\log(5) + x\log(7) &= \log(3) + (x-7)\log(2) \\
\log(5) + x\log(7) &= \log(3) + x\log(2) - 7\log(2) \\
\log(5) + x\log(7) - x\log(7) - \log(3) + 7\log(2) &= \log(3) + x\log(2) - 7\log(2) - x\log(7) - \log(3) + 7\log(2) \\
\log(5) - \log(3) + 7\log(2) &= x\log(2) - x\log(7) \\
\log(5) - \log(3) + 7\log(2) &= x(\log(2) - \log(7)) \\
\frac{\log(5) - \log(3) + 7\log(2)}{\log(2) - \log(7)} &= x \\
\frac{\log\left(\frac{5 \cdot 7^2}{3}\right)}{\log\left(\frac{2}{7}\right)} &= x \\
\frac{\log\left(\frac{245}{3}\right)}{\log\left(\frac{2}{7}\right)} &= x
\end{aligned}$$

The solution set is $\left\{\frac{\log\left(\frac{245}{3}\right)}{\log\left(\frac{2}{7}\right)}\right\}$. This may be simplified in an appropriate equivalent form, such as $\left\{\frac{\log\left(\frac{3}{245}\right)}{\log\left(\frac{7}{2}\right)}\right\}$ or $\left\{\frac{\ln\left(\frac{245}{3}\right)}{\ln\left(\frac{2}{7}\right)}\right\}$.

(g) $\log_4(2x+1) = 2$

$$\begin{aligned}
\log_4(2x+1) &= 2 \\
2x + 1 &= 4^2 \\
2x + 1 &= 16 \\
2x &= 15 \\
x &= \frac{15}{2}
\end{aligned}$$

Check:

$$\begin{aligned}
\log_4\left(2 \cdot \frac{15}{2} + 1\right) &= \log_4(16) \\
&= 2 \checkmark
\end{aligned}$$

The solution set is $\left\{\frac{15}{2}\right\}$.

(h) $\log_2(x) + \log_2(3) = \log_2(2)$

$$\begin{aligned} \log_2(x) + \log_2(3) &= \log_2(2) \\ \log_2(3x) &= \log_2(2) \\ 3x &= 2 \\ x &= \frac{2}{3} \end{aligned}$$

Check: $\log_2\left(\frac{2}{3}\right) + \log_2(3) = \log_2(2)$ ✓
The solution set is $\left\{\frac{2}{3}\right\}$.

(i) $\log_2(x) - \log_2(3) = \log_2(2)$

$$\begin{aligned} \log_2(x) - \log_2(3) &= \log_2(2) \\ \log_2\left(\frac{x}{3}\right) &= \log_2(2) \\ \frac{x}{3} &= 2 \\ x &= 6 \end{aligned}$$

Check: $\log_2(6) - \log_2(3) = \log_2(2)$ ✓
The solution set is $\{6\}$.

(j) $2\log_5(x-6) = \log_5(x)$

$$\begin{aligned} 2\log_5(x-6) &= \log_5(x) \\ \log_5(x-6)^2 &= \log_5(x) \\ (x-6)^2 &= x \\ x^2 - 12x + 36 &= x \\ x^2 - 13x + 36 &= 0 \\ (x-9)(x-4) &= 0 \\ x = 9 &\text{ or } x = 4 \end{aligned}$$

Check $x = 9$:

$$\begin{aligned} 2\log_5(9-6) &= 2\log_5(3) \\ &= \log_5(9) \checkmark \end{aligned}$$

Check $x = 4$: $2\log_5(4-6)$ is undefined.

The solution set is $\{9\}$.

(k) $\log_x\left(\sqrt{3}\right) = \frac{1}{4}$

$$\begin{aligned} \log_x\left(\sqrt{3}\right) &= \frac{1}{4} \\ x^{1/4} &= \sqrt{3} \\ x &= \left(\sqrt{3}\right)^4 \\ x &= 9 \end{aligned}$$

Check:

$$\begin{aligned}\log_9\left(\sqrt{3}\right) &= \log_9\left(3^{1/2}\right)\\ &= \frac{1}{2}\log_9(3)\\ &= \frac{1}{2}\cdot\frac{1}{2}\\ &= \frac{1}{4}\checkmark\end{aligned}$$

The solution set is $\{9\}$.

(l) $\log(1-x) = 2 + \log(1+x)$

$$\begin{aligned}\log(1-x) &= 2 + \log(1+x)\\ \log(1-x) - \log(1+x) &= 2\\ \log\left(\frac{1-x}{1+x}\right) &= 2\\ 10^2 &= \frac{1-x}{1+x}\\ 100 &= \frac{1-x}{1+x}\\ 100(1+x) &= 1-x\\ 100 + 100x &= 1-x\\ 99 &= -101x\\ -\frac{99}{101} &= x\\ x &\approx -0.98\end{aligned}$$

Check:
Left-hand side: $\log\left(1-\left(-\frac{99}{101}\right)\right) = \log\left(\frac{200}{101}\right)$
Right-hand side:

$$\begin{aligned}2+\log\left(1+\left(-\frac{99}{101}\right)\right) &= 2+\log\left(\frac{2}{101}\right)\\ &= \log\left(10^2\right)+\log\left(\frac{2}{101}\right)\\ &= \log\left(100\cdot\frac{2}{101}\right)\\ &= \log\left(\frac{200}{101}\right)\end{aligned}$$

The solution set is $\left\{-\frac{99}{101}\right\}$.

(m) $\log_6(x+4)+\log_6(x+3)=1$

$$\begin{aligned}\log_6(x+4)+\log_6(x+3)&=1\\ \log_6\big((x+4)(x+3)\big)&=1\\ (x+4)(x+3)&=6^1\\ x^2+7x+12&=6\\ x^2+7x+6&=0\\ (x+6)(x+1)&=0\\ x+6=0 &\text{ or } x+1=0\\ x=-6 &\text{ or } x=-1\end{aligned}$$

Check:
As $\log_6(-6+4)+\log_6(-6+3)$ is undefined, -6 is a false solution.

$$\begin{aligned}\log_6(-1+4)+\log_6(-1+3)&\stackrel{?}{=}1\\ \log_6(3)+\log_6(2)&\stackrel{?}{=}1\\ \log_6(3\cdot 2)&\stackrel{?}{=}\\ \log_6(6)&=1\checkmark\end{aligned}$$

The solution set is $\{-1\}$.

14. The percentage of carbon 14, Q, remaining in a fossil t years since decay began can be modeled by the function

$$Q=f(t)=100e^{-0.000124t}$$

(a) If a piece of cloth is thought to be 750 years old. What percentage of carbon 14 is expected to remain in this sample?

We will evaluate $f(750)$:

$$\begin{aligned}f(750)&=100e^{-0.000124(750)}\\ &\approx 91.12\end{aligned}$$

The cloth should have about 91.12% of its carbon 14 remaining.

(b) If a fossilized leaf contains 70% of its original carbon 14, how old is the fossil?

We will solve $f(t)=70$:

$$\begin{aligned}70&=100e^{-0.000124t}\\ 0.7&=e^{-0.000124t}\\ \ln(0.7)&=-0.000124t\\ \frac{\ln(0.7)}{-0.000124}&=t\\ t&\approx 2876.41\end{aligned}$$

The fossil is about 2876.41 years old.

15. The temperature of a cup of tea after it was brewed can be modeled by the function $T(t) = 100e^{-0.1t} + 68$, where t is the number of minutes since the tea was brewed and $T(t)$ is the temperature in degrees Fahrenheit at time t.

(a) Find and interpret $T(0)$.

$$\begin{aligned} T(0) &= 100e^{-0.01(0)} + 68 \\ &= 168 \end{aligned}$$

The initial temperature of the tea is 168^oF.

(b) Find and interpret $T(10)$.

$$\begin{aligned} T(10) &= 100e^{-0.1(10)} + 68 \\ &= 100e^{-1} + 68 \\ &\approx 104.8 \end{aligned}$$

After 10 minutes, the temperature of the tea is approximately 104.8^oF.

(c) Solve and interpret $T(t) = 80$.

$$\begin{aligned} T(t) &= 80 \\ 100e^{-0.1t} + 68 &= 80 \\ 100e^{-0.1t} &= 12 \\ e^{-0.1t} &= \frac{12}{100} \\ e^{-0.1t} &= \frac{3}{25} \\ -0.1t &= ln\left(\frac{3}{25}\right) \\ t &= \frac{ln\left(\frac{3}{25}\right)}{-0.1} \\ t &= -10\ln\left(\frac{3}{25}\right) \\ t &\approx 21.2 \end{aligned}$$

It takes approximately 21.2 minutes for the tea to reach 80^oF.

(d) Graph $y = T(t)$ in your calculator. What is the horizontal asymptote?

The horizontal asymptote is $y = 68$. (This happens to be the room temperature.)

16. Tom and Jerry make separate investments at the same time. Their respective investments can be modeled by the functions

$$T = f(t) = 5000(1.065)^t \qquad \text{and} \qquad J = g(t) = 4500\left(1 + \frac{0.065}{12}\right)^{12t}$$

where t is the number of years since each investment began and T and J are their respective investment values in dollars.

(a) How much does Tom invest initially? How much does Jerry invest initially?

Tom's initial investment was $\$5,000$. Jerry's initial investment was $\$4,500$.

(b) What will the values of their respective investments be after 5 years?

Tom's investment after 5 years:

$$\begin{aligned} f(5) &= 5000(1.065)^5 \\ &\approx 6850.43 \end{aligned}$$

Jerry's investment after 5 years:

$$\begin{aligned} g(5) &= 4500\left(1 + \frac{0.065}{12}\right)^{12(5)} \\ &= 6222.68 \end{aligned}$$

After 5 years, Tom's investment will be worth \$6850.43 and Jerry's investment will be worth \$6222.68.

(c) How long will it take for Jerry's investment to double?

As Jerry's initial investment is $\$4,500$, we will solve $g(t) = 9000$:

$$\begin{aligned} 9000 &= 4500\left(1 + \frac{0.065}{12}\right)^{12t} \\ 2 &= \left(1 + \frac{0.065}{12}\right)^{12t} \\ \ln(2) &= \ln\left(\left(1 + \frac{0.065}{12}\right)^{12t}\right) \\ \ln(2) &= 12t\ln\left(1 + \frac{0.065}{12}\right) \\ \frac{\ln(2)}{12\ln\left(1 + \frac{0.065}{12}\right)} &= t \\ t &\approx 10.69 \end{aligned}$$

Jerry's investment will double in approximately 10.69 years.

17. The acidity of a solution is measured using the pH scale. Lucky for you, it's a logarithmic scale! The pH depends on the concentration, or molarity, in moles/liter of hydrogen ions. The formula for pH is given by $\text{pH} = -\log\left(\left[\text{H}^+\right]\right)$, where $[\text{H}^+]$ is the molarity of hydrogen ions. A low pH is very acidic (like lemons) and a high pH is very basic (like bleach).

(a) Coffee has a hydrogen ion concentration of about $\left[\text{H}^+\right] = 1.1 \times 10^{-5}$. Find the pH of coffee.

$$\begin{aligned} \text{pH} &= -\log\left(1.1 \times 10^{-5}\right) \\ &\approx 4.959 \end{aligned}$$

The pH of coffee is about 4.959.

(b) Pure water is called neutral and has a pH of 7. Find the hydrogen ion concentration of pure water.

$$\begin{aligned} 7 &= -\log\left(\left[\text{H}^+\right]\right) \\ -7 &= \log\left(\left[\text{H}^+\right]\right) \\ 10^{-7} &= \left[\text{H}^+\right] \end{aligned}$$

The hydrogen ion concentration of pure water is 1×10^{-7} moles per liter.

(c) The hydrogen ion concentration of bleach is about $\left[\text{H}^+\right] = 3.16 \times 10^{-13}$. Find the pH of bleach.

$$\begin{aligned} \text{pH} &= -\log\left(3.16 \times 10^{-13}\right) \\ &\approx 12.500 \end{aligned}$$

The pH of bleach is about 12.500.

(d) The pH of pure Hydrochloric acid (HCl) is 0. Find the hydrogen ion concentration of HCl.

$$\begin{aligned} 0 &= -\log\left(\left[\text{H}^+\right]\right) \\ 0 &= \log\left(\left[\text{H}^+\right]\right) \\ 10^{0} &= \left[\text{H}^+\right] \\ 1 &= \left[\text{H}^+\right] \end{aligned}$$

The hydrogen ion concentration of pure HCl is 1 mole per liter.

18. Let $g(x) = x^4 - 3x^3 + 4x$.

(a) Use your calculator to find the zeros of the function.
The zeros of g are -1, 0 and 2.

(b) Use your calculator to estimate the interval where the graph is concave down. Round accurately to the nearest tenth.
The graph is concave down over the approximate interval $(0, 1.5)$.

(c) Use your calculator to estimate the local maximum value and where it occurs.
The local maximum value is approximately 2.08 and occurs at approximately 0.84.

(d) Use your calculator to estimate the absolute minimum value and where it occurs.
The absolute minimum value is approximately -1.62 and occurs at approximately -0.59.

19. Find possible formulas for each polynomial function in Figure 7. Clearly list the zeros and their multiplicities.

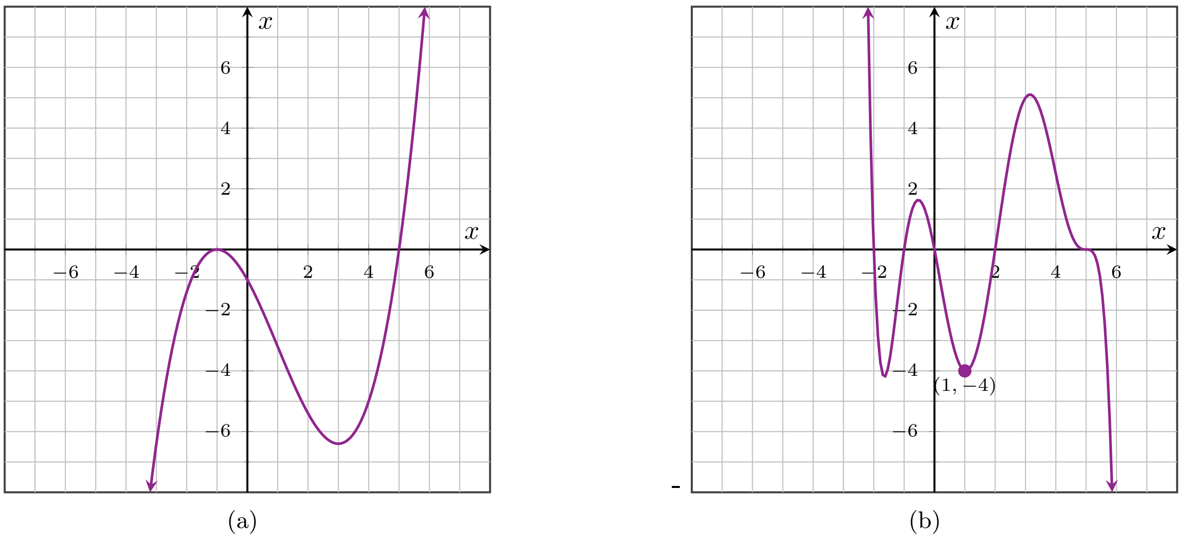


FIGURE 7

Solution to Figure 7(a)

The zeros of f are -1 and 5.

zero	multiplicity	factor
-1	even: 2	$(x+1)^2$
5	odd: 1	$(x-5)$

A possible formula for this function is of the form $f(x) = k(x+1)^2(x-5)$. We will use the point $(0, -1)$ to determine k. Note that any point *exactly* on the graph of $y = f(x)$ could be used.

$$-1 = k(0+1)^2(0-5)$$
$$\frac{1}{5} = k$$

A possible formula for this polynomial function is $f(x) = \frac{1}{5}(x+1)^2(x-5)$.

Solution to Figure 7(b)

The zeros of f are -2, -1, 0, 2, and 5.

zero	multiplicity	factor
-2	odd: 1	$(x+2)$
-1	odd: 1	$(x+1)$
0	odd: 1	(x)
2	odd: 1	$(x-2)$
5	odd: 3	$(x-5)^3$

A possible formula for this function is of the form $f(x) = kx(x+2)(x+1)(x-2)(x-5)^3$. We will use the point $(1, -4)$ to find k:

$$-4 = k(1)(1+2)(1+1)(1-2)(1-5)^3$$
$$-4 = k(3)(2)(-1)(-64)$$
$$k = -\frac{1}{96}$$

A possible formula for this polynomial function is $f(x) = -\frac{1}{96}x(x+2)(x+1)(x-2)(x-5)^3$.

20. Sketch a graph of $y = f(x)$ for each polynomial function below. Also list the zeros and their multiplicity, the vertical intercept, and the long-run behavior.

(a) $f(x) = -x^2(x+4)$

- Zeros
  - The zero at $(0,0)$ repeats twice. Thus the function will "bounce" at this point.
  - The zero at $(-4,0)$ does not repeat. Thus the function will go straight through the horizontal axis at this point.
- Vertical Intercept
  - The vertical intercept is $(0,0)$.
- Long-run Behavior
  - This is a third-degree polynomial function with a negative leading coefficient.
  - As $x \to \infty$, $f(x) \to -\infty$
  - As $x \to -\infty$, $f(x) \to \infty$

The graph is shown in Figure 8.

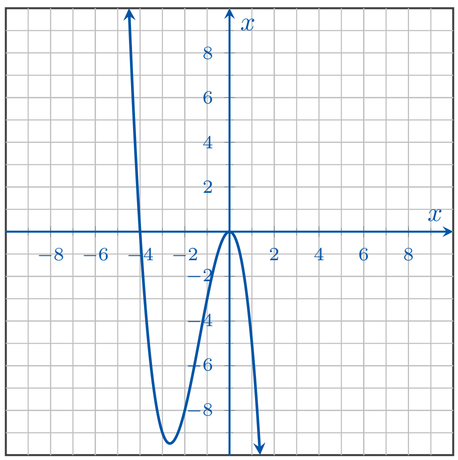


FIGURE 8

(b) $g(x) = (x-2)(x+1)^2(x+2)$

- Zeros
  - The zero at $(2, 0)$ does not repeat. Thus the function will go straight through the horizontal axis at this point.
  - The zero at $(-1, 0)$ repeats twice. Thus the function will "bounce" at this point.
  - The zero at $(-2, 0)$ does not repeat. Thus the function will go straight through the horizontal axis at this point.
- Vertical Intercept
  - The vertical intercept is $(0, -4)$.
- Long-run Behavior
  - This is a fourth-degree polynomial function with a positive leading coefficient.
  - As $x \to \infty$, $f(x) \to \infty$
  - As $x \to -\infty$, $f(x) \to \infty$

The graph is shown in Figure 9.

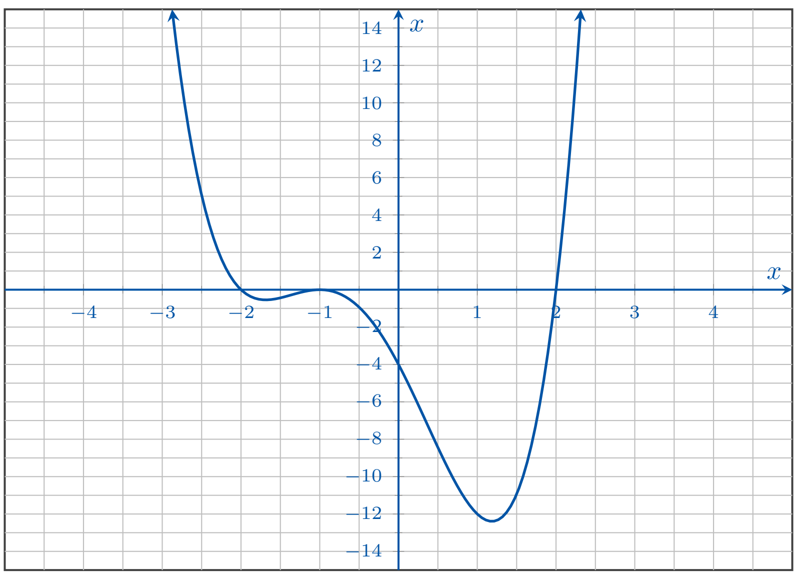
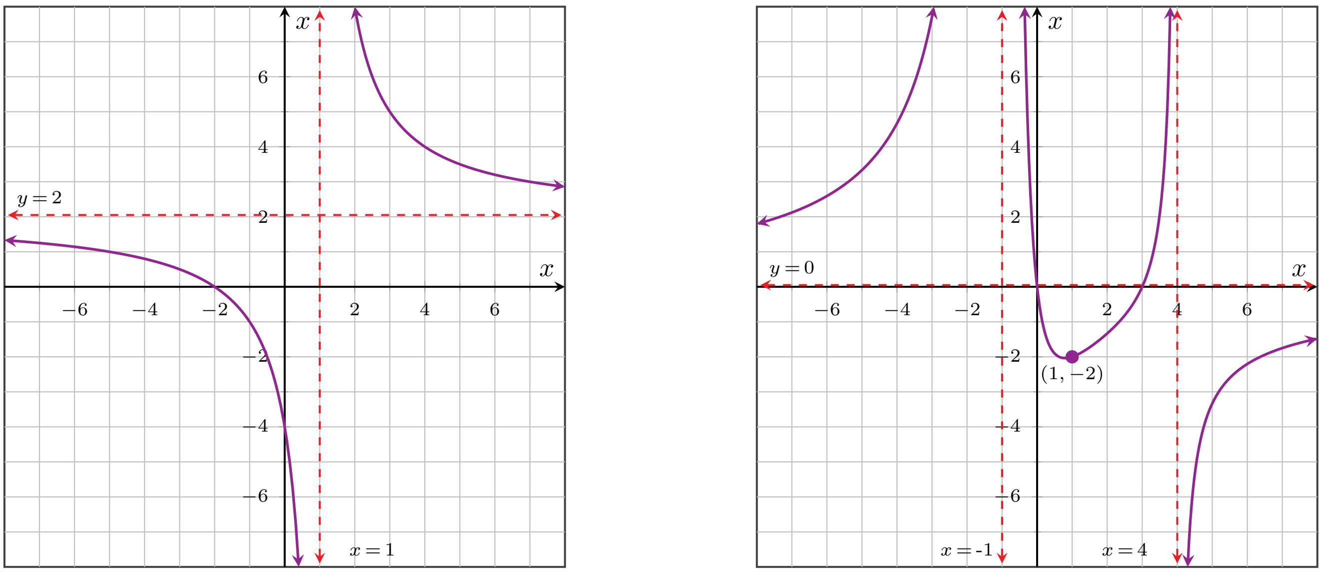


Figure 9

21. Find possible formulas for each rational function in Figure 11 below. List the zeros and their multiplicity, any vertical asymptotes and any horizontal asymptotes.



(a) (b)

FIGURE 10

Solution to Figure 11(a)

TABLE 3. Behavior of R

Zeros/ VA	Multiplicity	Factor	Numerator/denominator
zero: -2	odd (1)	$(x+2)$	numerator
VA: $x=1$	odd (1)	$(x-1)$	denominator

Thus a possible formula for $R(x)$ is of the form

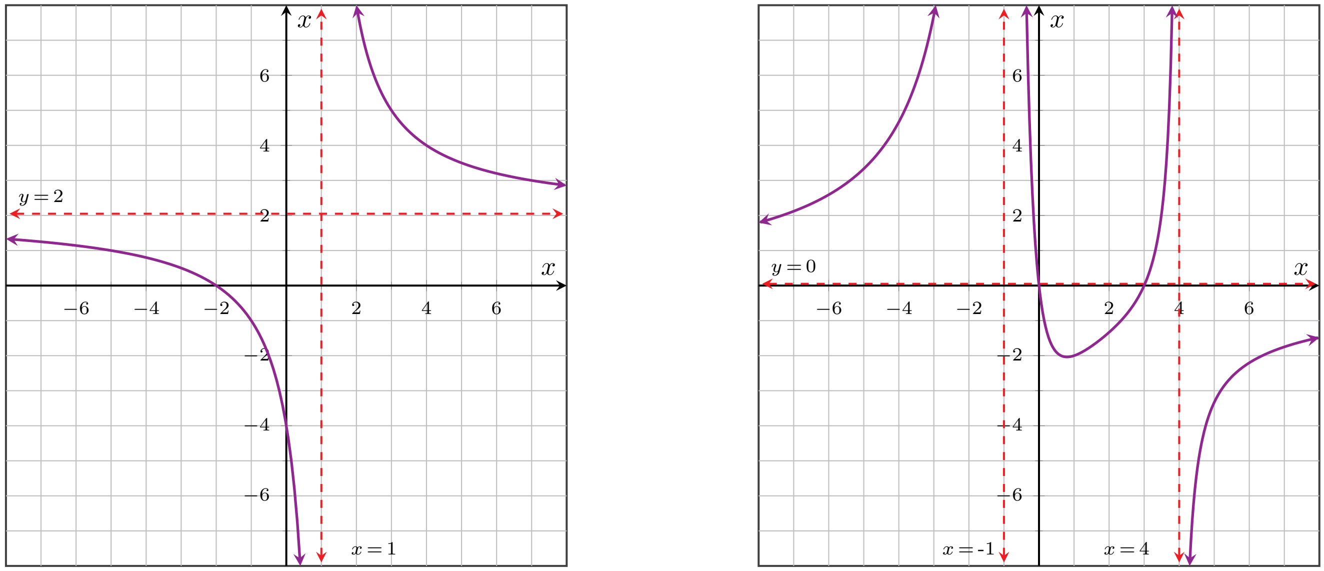
$$R(x) = \frac{k(x+2)}{(x-1)}$$

where k is a constant factor.
The constant factor of k can be determined to be 2 based upon the horizontal asymptote $y=2$.
Therefore a possible formula for $R(x)$ is

$$R(x) = \frac{2(x+2)}{(x-1)}$$

Alternately, we could determine k by using one point on the graph that is not a horizontal intercept. Here, the point $(0,-4)$ is given:

$$R(0) = -4$$
$$\frac{k(0+2)}{(0-1)} = -4$$
$$k = 2$$



(a) (b)

FIGURE 11

Solution to Figure 11(b)

TABLE 4. Behavior of R

Zeros/ VA	Multiplicity	Factor	Numerator/denominator
zero: 0	odd (1)	(x)	numerator
zero: 3	odd (1)	$(x-3)$	numerator
VA: $x=-1$	even (2)	$(x+1)^2$	denominator
VA: $x=4$	odd (1)	$(x-4)$	denominator

Thus a possible formula for $R(x)$ is of the form

$$R(x) = \frac{kx(x-3)}{(x+1)^2(x-4)}$$

where k is a constant factor.

Since the horizontal asymptote is $y=0$, we cannot use it to determine k. We can use the point $(1,-2)$ to determine it instead:

The constant factor of k can be determined to be 2 based upon the horizontal asymptote $y=2$.

$$\begin{aligned} -2 &= \frac{k(1)(1-3)}{(1+1)^2(1-4)} \\ -2 &= \frac{k(-2)}{-12} \\ k &= -12 \end{aligned}$$

Therefore a possible formula for $R(x)$ is

$$R(x) = \frac{-12x(x-3)}{(x+1)^2(x-4)}$$

22. Find a possible formula for the rational function in Figure 12 below. List the zeros and their multiplicity, any vertical asymptotes and any horizontal asymptotes.

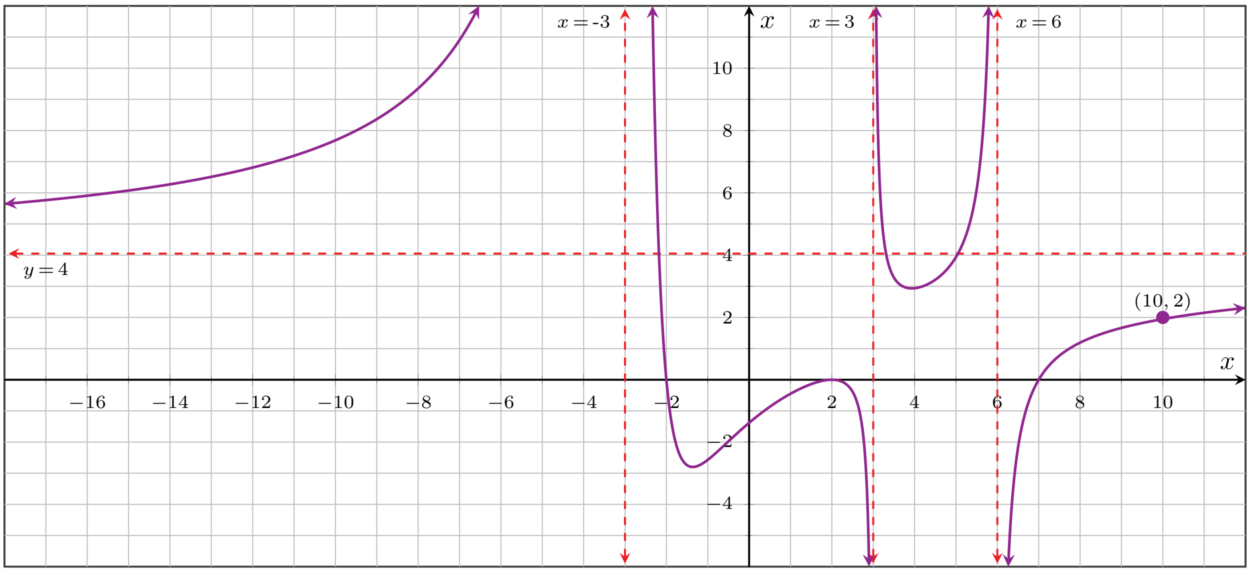


FIGURE 12

Solution to Figure 12

TABLE 5. Behavior of R

Zeros/ VA	Multiplicity	Factor	Numerator/denominator
zero: -2	odd (1)	$(x+2)$	numerator
zero: 2	even (2)	$(x-2)^2$	numerator
zero: 7	odd (1)	$(x-7)$	numerator
VA: $x=-3$	even (2)	$(x+3)^2$	denominator
VA: $x=3$	odd (1)	$(x-3)$	denominator
VA: $x=6$	odd (1)	$(x-6)$	denominator

Thus a possible formula for $R(x)$ is of the form

$$R(x) = \frac{k(x+2)(x-2)^2(x-7)}{(x+3)^2(x-3)(x-6)}$$

where k is a constant factor.
The constant factor of k can be determined to be 4 based upon the horizontal asymptote $y=4$.
Therefore a possible formula for $R(x)$ is

$$R(x) = \frac{4(x+2)(x-2)^2(x-7)}{(x+3)^2(x-3)(x-6)}$$